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CONSTRUCTION OF SECOND ORDER ROTATABLE DESIGNS ASSOCIATED WITH PARTIALLY BALANCED INCOMPLETE BLOCK (PBIB) DESIGNS

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SUMMARY

A new method of constructing second order rotatable design (SORD) using two suitably chosen partially balanced incomplete block (PBIB) designs is suggested in this paper.

Keywords: Second Order Rotatable Designs; Partially Balanced incomplete block Designs.

1. Introduction

Rotatable designs were first introduced by Box and Hunter [1]. They obtained these designs through geometrical configurations. Later Bose and Draper [3], Das and Narasimhan [4], Tyagi [6] and others suggested different techniques for construction of second order rotatable designs. Gupta [5] also constructed second order response surface design with the help of two associate classes PBIB designs with lesser number of design points. But these designs are not rotatable. We have shown in this paper that the conditions of rotatibility can be satisfied for the design points obtained by combining two suitably chosen such SORS designs.

2. Method of Construction

For the construction of SORD, Das and Narasimham [4] have taken incidence matrix of a BIBD after replacing 1 by some unknown level α

and multiplying this magnitude set with those of a 2^k factorial with levels denoted by $+1$ and -1. If $k > 4$, instead of taking all the combinations a proper fraction can be used such that no main effect or interactions with less than five factors get confounded.

Gupta [5] developed SORS designs by selecting magnitude set as incidence matrix of a PBIB designs replacing 1 by some unknown level α. These design-points gave the following results:

$$\text{A}: \quad \sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \quad \text{if any of the } \alpha_i \text{ is odd for}$$
$$\alpha_i' = 0, 1, 2 \text{ or } 3 \text{ and } \Sigma\alpha_i \leqslant 4$$

$$\text{B}: \text{(i)} \quad \sum_{u=1}^{N} x_{iu}^2 = N\lambda_2 = r\alpha^2 2^k$$

$$\text{(ii)} \quad \sum_{u=1}^{N} x_{iu}^4 = CN\lambda_4 = r\alpha^4 2^k \text{ for all } i$$

$$\text{C}: \text{(i)} \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \lambda_1^1 \alpha^4 2^k \text{ if } (i, j) \in S_1$$

$$\text{(ii)} \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \lambda_2^1 \alpha^4 2^k \text{ if } (i, j) \in S_2$$

where, λ_1^1, λ_2' are the parameters of PBIB design. For rotatability we require to satisfy the condition

$$\text{D}: \quad \sum_{u=1}^{N} x_{iu}^4 = 3 \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2,$$

which is not satisfied in these SORS designs.

For a PBIB design with two association classes, let S_1 be the set of pairs of treatment which occur together in λ_1 blocks. The number of pairs in S_1 is $vn_1/2$. Let the remaining pairs of treatments belong to S_2 where each of the pair will occur together in λ_2 blocks and number of such pairs is $vn_2/2$. We shall call two PBIB with similar association a scheme if the sets S_1 and S_2 remain unaltered but the values of λ's may be different. Now, if we take the incidence matrix of another PBIB design similar to the first one with values of λ as λ_1^2 and λ_2^2 and replacing 1 by β, we shall get another set of N_2 points by multiplying with suitable uneffected set of combinations (U.S.C.).

The totality of $(N_1 + N_2)$ design points will satisfy the following conditions:

$$\text{A}: \quad \sum_{u=1}^{N} \prod_{i=1}^{v} x_{iu}^{\alpha_i} = 0 \text{ if any of the } \alpha_i \text{ is odd for}$$

$$\alpha_i = 0, 1, 2 \text{ or } 3 \text{ and } \Sigma\alpha_i \leqslant 4$$

$$\text{B}: \text{(i)} \quad \sum_{u=1}^{N} x_{iu}^2 = r_I 2^{k_I} \alpha^2 + r_2 2^{k_2} \beta^2 = (N_I + N_2)\,\lambda_2 = \text{constant}$$

$$\text{(ii)} \quad \sum_{u=1}^{N} x_{iu}^4 = r_I 2\lambda\alpha^4 + r_2 2^{k_2}\,\beta^4$$

$$= C(N_1 + N_2)\,\lambda_4 = \text{constant.}$$

$$\text{C}: \text{(i)} \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \lambda_I' 2^{k_I} \alpha^4 + \lambda_2^2 2^{k_2} \beta^4 \text{ for } (i, j) \in s_I$$

$$\text{(ii)} \quad \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \lambda_2' 2^{k_I} \alpha^4 + \lambda_2^2 2^{k_2} \beta^4 \text{ for } (i, j) \in s_2$$

where λ_I^2 and λ_2^2, are the parameters of second PBIB design. Now, if λ and β are so chosen that

$$\lambda_I' 2^{k_I} \alpha^4 + \lambda_I^2 2^{k_2} \beta^4 = \lambda_2' 2^{k_I}\,\alpha^4 + \lambda_2^2 2^{k_2}\,\beta^4$$

then we get

$$\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = \text{constant.}$$

Hence, we have the condition

$$\text{D}: \quad \sum_{u=1}^{N} x_{iu}^4 = C \sum_{u=1}^{N} x_{iu}^2 x_{ju}^2.$$

For rotatibility the value of C should be equal to 3.

Now if $C = 3$, either $2v$ $(= N_3)$ points of the type

$$(\pm\,\gamma, 0, \ldots, 0)$$
$$(0, 0, \ldots, \pm\,\gamma)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$(0, 0, \ldots, \pm\,\gamma)$$

When $C < 3$ or fraction of 2^v points $(= N_3)$ obtained from the magnitude set of the type $(\gamma, \gamma, \ldots, \gamma)$ when $C > 3$ are added to the design points.

The value of γ may be fixed such that the condition $C = 3$ is satisfied. If the totality of the points does not satisfy the condition

$$E : \frac{\lambda_4}{\lambda_2^2} > \frac{\nu}{\nu + 2},$$

then some central points need to be added.

Example. Let us consider two PBIB designs with $\nu = 6$ and other parameters are:

$$b_1 = 3,\ r_1 = 1,\ K_1 = 2,\ \lambda_1 = 0,\ \lambda_2' = 1. \qquad (1)$$
$$(3, 6) \quad (2, 5) \quad (1, 4)$$

$$b_2 = 4,\ r_2 = 2,\ K_2 = 3,\ \lambda_1^2 = 1,\ \lambda_2^2 = 0 \qquad (2)$$
$$(1, 2, 3)\ (1, 5, 6)\ (2, 4, 6)\ (3, 4, 5).$$

We have $N_1 = 12$ and $N_2 = 32$.

For the value of $\alpha^4 = 2\,\beta^4$, the 44 designs points for 6 factors satisfy all the conditions of rotatibility.

Analysis. The estimate of β's of the second order response surface

$$Y_u = \beta_o + \sum_{i=1}^{\nu} \beta_i x_{iu} + \sum_{i \leqslant j=1}^{\nu} \beta_{ij} x_{iu}^2 + \sum_{i \leqslant j \leqslant k=1}^{\nu} \beta_{ijk} x_{iu} x_{ju} x_{ku} + \ldots, \qquad i = 1, 2, \ldots, 6$$

for 6 factors in 44 points are obtained as follows:

$$b_0 = \frac{0.7272\ \Sigma\ yu - 0.348\ \Sigma\Sigma\ xiu^2 yu}{0.0264}$$

$$b_i = 0.065\ \Sigma\ x_{iu} y_u$$
$$b_{ij} = 0.25\ \Sigma\ x_{iu} y_u$$
$$b_{ij} = 0.125\ \Sigma\ x_{iu}^2\ y_u - 1.318\ \Sigma\ y_u + 6.3\ \Sigma\Sigma\ x_{iu}^2\ y_u$$

and the estimated variance are:

$$V(b_0) = 27.54\ \sigma^2$$
$$V(b_i) = 0.0653\ \sigma^2$$
$$V(b_{ij}) = 0.25\ \sigma^2$$
$$V(b_{ii}) = 6.425\ \sigma^2$$
$$\text{Cov.}\ (b_0, b_{ii}) = -1.314\ \sigma^2$$
$$\text{Cov.}\ (b_{ii} b_{jj}) = 6.3\ \sigma^2$$

Y_u being the estimated response corresponding to the point (X_{iu}, X_{2u}, . . . , X_{6u})

If $\Sigma\, x_{iu}^2 = d^2$, then

$$V(Y_u) = (27.54 - 2.562\, d^2 + 6.425\, d^4)\, \sigma^2.$$

Rotatable designs with 6 factors reported by Das and Narasimham [4] were obtained through the BIB design with the parameters

$$V = 6,\ b = 10,\ r = 5,\ K = 3,\ \lambda = 2$$

in 92 design points and the variance of estimated response for the corresponding point is $V(Y_u) = [2.35 - 0.585\, d^2 + 1.235\, d^4]\, \sigma^2$.

Let us consider the designs obtained through PBIB design as type-I and that through BIB design as type-II, we worked out the relative efficiency of type-I design as compared to type-II by the method as suggested by Box and Wilson [2].

If $V(Y_u)$ be the variance of estimated response at the point (x_{1u}. x_{2u}, . . . , x_{vu}) $\left[\sum_{i=1}^{v} x_{iu}^2 = d^2\right]$, then information available per design point is $1/NV(y_u)$, where N is the number of design points, the relative efficiency has been worked out as the ratio of information per point for the two designs. This ratio will be a function of d^2 if we assume the value of σ^2 remains same for both the designs. For different values of d, the relative efficiency of type-I design compared to type-II has been given in Table 1.

TABLE 1

Distance of points from the origin	*Relative efficiency*
0.0	0.1784
0.1	0.1781
0.2	0.1774
0.3	0.1763
0.4	0.1753
0.5	0.1747
0.6	0.1752
0.7	0.1773
0.8	0.1819
0.9	0.1893
1.0	0.1998

In the present study type-I designs in 44 design points have been obtained. If however, few central points are added, then the minimum relative efficiency of the design will increase considerably. The relative efficiency of type-I design for 6 factors in 50 points (adding six more central points) with type-II designs in 92 points has been worked out and given in Table 2.

TABLE 2

Distance of points from the origin	*Relative efficiency*
0.0	26.2857
0.1	26.2206
0.2	26.6258
0.3	27.1352
0.4	27.7043
0.5	28.5454
0.6	29.5865
0.7	30.7553
0.8	31.9283
0.9	32.9228
1.0	17.7568

Thus the minimum relative efficiency of the design obtained through PBIB designs for 6 factors in 50 points lies between 17.76 and 32.92 as compared to type-II rotatable design where the design points required is as high as 92 points.

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